

TRANSPORT TECHNIQUES FOR NON-GAUSSIANITY

DAVID J. MULRYNE

*School of Physics and Astronomy, Queen Mary University of London,
London, E1 4NS, United Kingdom*

This proceedings contribution provides a brief update on the transport approach to calculating the statistics of perturbations produced during inflation. It is based in particular on work in collaboration with Gemma Anderson and David Seery.¹

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1. Inflationary statistics

During inflation, quantum fluctuations become classical perturbations when their wavelengths becomes larger than the cosmological horizon. Different models of inflation produce perturbations with different statistical properties, and we wish to track the evolution of these statistics. Transport techniques^{1–3} are one method to do this. They offer analytic insights into the evolution,^{4,5} and reduce the problem to a set of coupled ordinary differential equations that can easily be solved numerically.

For most applications we are interested in just the first few cumulants (inflationary correlation functions) of the Fourier modes of the curvature perturbation ζ . These are related in a straightforward manner to the statistics of the field perturbations, which in multi-field canonical models are very close to Gaussian at horizon crossing⁶ but evolve thereafter. In these proceedings, we review how this super-horizon evolution of statistics can be calculated.

2. Evolution of field space statistics and ‘shapes’

We begin with the evolution equation for the Fourier modes of the field fluctuations themselves. These can be written in a DeWitt index form^{1,5} in which a compound primed index includes a field label α and a momentum label \mathbf{k}_α . The summation convention applied to α' implies integration over momentum with measure $d^3\mathbf{k}_\alpha/(2\pi)^3$, and summation over the field species. The equations are

$$\begin{aligned} \frac{d\delta\varphi_{\alpha'}}{dN} = & u_{\alpha'\beta'}\delta\varphi_{\beta'} + \frac{1}{2!}u_{\alpha'\beta'\gamma'}\left(\delta\varphi_{\beta'}\delta\varphi_{\gamma'} - \langle\delta\varphi_{\beta'}\delta\varphi_{\gamma'}\rangle\right) \\ & + \frac{1}{3!}u_{\alpha'\beta'\gamma'\delta'}\left(\delta\varphi_{\beta'}\delta\varphi_{\gamma'}\delta\varphi_{\delta'} - \langle\delta\varphi_{\beta'}\delta\varphi_{\gamma'}\delta\varphi_{\delta'}\rangle\right) + \dots, \quad (1) \end{aligned}$$

where the time variable N is e-folds. The u coefficients take the form $u_{\alpha'\beta'} \equiv (2\pi)^3\delta(\mathbf{k}_\alpha - \mathbf{k}_\beta)u_{\alpha\beta}[N]$, $u_{\alpha'\beta'\gamma'} \equiv (2\pi)^3\delta(\mathbf{k}_\alpha - \mathbf{k}_\beta - \mathbf{k}_\gamma)u_{\alpha\beta\gamma}[N]$ and $u_{\alpha'\beta'\gamma'\delta'} \equiv (2\pi)^3\delta(\mathbf{k}_\alpha - \mathbf{k}_\beta - \mathbf{k}_\gamma - \mathbf{k}_\delta)u_{\alpha\beta\gamma\delta}[N]$, where u with unprimed indices is just a function of unperturbed background quantities, and hence of N . Here we have assumed slow-roll and so neglected field velocity perturbations (this can be easily relaxed).

Utilising the simple principal $d\langle A \rangle/dN = \langle dA/dN \rangle$, which follows from probability conservation,⁵ and identifying A with products of field perturbations, we

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find

$$\begin{aligned}
\frac{d\Sigma_{\alpha'\beta'}}{dN} &= u_{\alpha'\gamma'}\Sigma_{\gamma'\beta'} + u_{\beta'\gamma'}\Sigma_{\gamma'\alpha'} + \dots, \\
\frac{d\alpha_{\alpha'\beta'\gamma'}}{dN} &= u_{\alpha'\lambda'}\alpha_{\lambda'\beta'\gamma'} + (u_{\alpha'\lambda'\mu'}\Sigma_{\lambda'\beta'}\Sigma_{\mu'\gamma'} + 3 \text{ cyclic}) + \dots, \\
\frac{d\beta_{\alpha'\beta'\gamma'\delta'}}{dN} &= \left(u_{\alpha'\lambda'}\beta_{\lambda'\beta'\gamma'\delta'} + 3 \text{ cyclic}\right) + \left(u_{\alpha'\lambda'\mu'}\alpha_{\lambda'\beta'\gamma'}\Sigma_{\mu'\delta'} + 11 \text{ cyclic}\right) \\
&\quad + \left(u_{\alpha'\lambda'\mu'\nu'}\Sigma_{\lambda'\beta'}\Sigma_{\mu'\gamma'}\Sigma_{\nu'\delta'} + 3 \text{ cyclic}\right) + \dots,
\end{aligned} \tag{2}$$

where the dots indicate each equation is truncated at leading order, and $\Sigma_{\alpha'\beta'} = \langle \delta\phi_{\alpha'}\delta\phi_{\beta'} \rangle$, $\alpha_{\alpha'\beta'\gamma'} = \langle \delta\phi_{\alpha'}\delta\phi_{\beta'}\delta\phi_{\gamma'} \rangle$ and $\beta_{\alpha'\beta'\gamma'\delta'} = \langle \delta\phi_{\alpha'}\delta\phi_{\beta'}\delta\phi_{\gamma'}\delta\phi_{\delta'} \rangle - \Sigma_{\alpha'\beta'}\Sigma_{\gamma'\delta'} - \Sigma_{\alpha'\gamma'}\Sigma_{\beta'\delta'} - \Sigma_{\alpha'\delta'}\Sigma_{\beta'\gamma'}$. These represent the equations of motion for cummulants/correlations of the field fluctuations.

On super-horizon scales, not all ‘shapes’ of correlations are generated. Starting with Gaussian fluctuations with close to scale invariant power spectrum, one finds¹

$$\begin{aligned}
\Sigma_{\alpha'\beta'} &= (2\pi)^3\delta(\mathbf{k}_\alpha + \mathbf{k}_\beta)\frac{\Sigma_{\alpha\beta}}{k_\alpha^3}, \\
\alpha_{\alpha'\beta'\gamma'} &= (2\pi)^3\delta(\mathbf{k}_\alpha + \mathbf{k}_\beta + \mathbf{k}_\gamma)\left(\frac{a_{\alpha|\beta\gamma}}{k_\beta^3k_\gamma^3} + \frac{a_{\beta|\alpha\gamma}}{k_\alpha^3k_\gamma^3} + \frac{a_{\gamma|\alpha\beta}}{k_\alpha^3k_\beta^3}\right), \\
\beta_{\alpha'\beta'\gamma'\delta'} &= (2\pi)^3\delta(\mathbf{k}_\alpha + \mathbf{k}_\beta + \mathbf{k}_\gamma + \mathbf{k}_\delta)\left(\frac{g_{\alpha|\beta\gamma\delta}}{k_\beta^3k_\gamma^3k_\delta^3} + 3 \text{ cyclic}\right. \\
&\quad \left. + \frac{\tau_{\alpha\beta|\gamma\delta}}{k_\alpha^3k_\beta^3|\mathbf{k}_\alpha + \mathbf{k}_\gamma|^3} + 11 \text{ cyclic}\right),
\end{aligned} \tag{3}$$

where $\Sigma_{\alpha\beta}$, $a_{\alpha|\beta\gamma}$, $g_{\alpha|\beta\gamma\delta}$ and $\tau_{\alpha\beta|\gamma\delta}$ carry only very mild scale dependence. These shape parameters obey their own transport equations¹

$$\begin{aligned}
\frac{da_{\alpha|\beta\gamma}}{dN} &= u_{\alpha\lambda}a_{\lambda|\beta\gamma} + u_{\beta\lambda}a_{\alpha|\lambda\gamma} + u_{\gamma\lambda}a_{\alpha|\beta\lambda} + u_{\alpha\lambda\mu}\Sigma_{\lambda\beta}\Sigma_{\mu\gamma}, \\
\frac{dg_{\alpha|\beta\gamma\delta}}{dN} &= u_{\alpha\lambda}g_{\lambda|\beta\gamma\delta} + (u_{\beta\lambda}g_{\alpha|\lambda\gamma\delta} + u_{\alpha\lambda\mu}a_{\lambda|\beta\gamma}\Sigma_{\mu\delta} + 3 \text{ cyclic}) + u_{\alpha\lambda\mu\nu}\Sigma_{\lambda\beta}\Sigma_{\mu\gamma}\Sigma_{\nu\delta}, \\
\frac{d\tau_{\alpha\beta|\gamma\delta}}{dN} &= u_{\alpha\lambda}\tau_{\lambda\beta|\gamma\delta} + u_{\beta\lambda}\tau_{\alpha\lambda|\gamma\delta} + u_{\gamma\lambda}\tau_{\alpha\beta|\lambda\delta} + u_{\delta\lambda}\tau_{\alpha\beta|\gamma\lambda} \\
&\quad + u_{\gamma\lambda\mu}\Sigma_{\mu\alpha}a_{\delta|\lambda\beta} + u_{\delta\lambda\mu}\Sigma_{\mu\beta}a_{\gamma|\lambda\alpha},
\end{aligned} \tag{4}$$

and can be used to form the correlation functions of ζ , and then calculate the related non-Gaussian parameters of ζ , namely f_{NL} , τ_{NL} and g_{NL} (see Refs. 7–9) as explained in detail in Ref. 1. These equations, therefore, represent a practical algorithm to evolve f_{NL} , τ_{NL} and g_{NL} from horizon crossing.

For example, numerically solving the equations for the hybrid potential model with two light fields,¹⁰ $V = M^4 \left[\frac{1}{2}m^2\phi_1^2 + \frac{1}{2}g^2\phi_1^2\phi_2^2 + \frac{\lambda}{4}(\phi_2^2 - v^2)^2 \right]$, with parameter choices and initial conditions $g^2 = v^2/\phi_{\text{crit}}^2$, $m^2 = v^2$, $v = 0.2M_{\text{pl}}$, $\phi_{\text{crit}} = 20M_{\text{pl}}$, $\lambda = 5$, $\phi_{1\text{exit}} = 15.5M_{\text{pl}}$ and $\phi_{2\text{exit}} = 0.0015M_{\text{pl}}$, results in the evolution of the non-Gaussianity parameters plotted in Fig. 1.

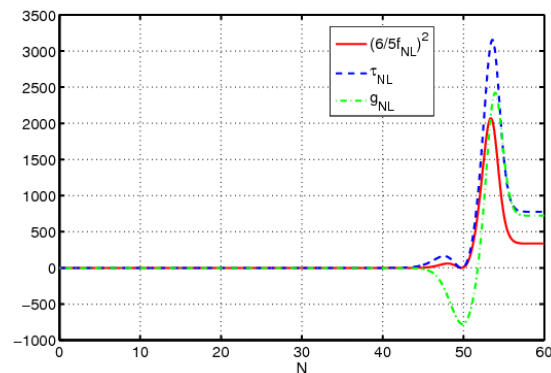


Fig. 1. The evolution of f_{NL} , τ_{NL} and g_{NL} from horizon-crossing to the end of inflation for the model of inflation defined in the text.

3. Discussion

Space has only permitted a brief review, so it is worth mentioning work on the transport approach to inflationary observables has yielded a number of other important results. In particular, we glossed over how the statistics of field perturbations are converted to those of ζ , and a particularly simple derivation of these relations can be found.¹ Moreover, the correlations involved in the transport method can be related to the coefficients of a δN^7 style Taylor expansion, leading to transport equations for the Taylor coefficients themselves.^{1,5} Furthermore, a geometrical decomposition of the transport equations is possible.⁵ Finally, current work is ongoing in extending the transport method to sub-horizon scales¹¹ and providing reusable numerical tools based on transport methods.

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